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Normal frames and the validity of the equivalence principle

II. The case along paths

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Abstract

We investigate the validity of the equivalence principle along paths in gravitational theories based on derivations of the tensor algebra over a differentiable manifold. We prove the existence of local bases, called normal, in which the components of the derivations vanish along arbitrary paths. All such bases are explicitly described. The holonomicity of the normal bases is considered. The results obtained are applied to the important case of linear connections and their relationship with the equivalence principle is described. In particular, any gravitational theory based on tensor derivations which obeys the equivalence principle along all paths, must be based on a linear connection.

1 Introduction

A well known classical result is the existence of local coordinates in which the components of a symmetric linear connection [1] vanish along a smooth path without self-intersections [2, 3]. For the first time it was observed by Fermi [4] for the Christoffel symbols of a Riemannian connection and later it was generalized for arbitrary symmetric linear connections [5, Sec. 25, pp. 64–68]. It is natural for these results to be generalized to the case of nonvanishing torsion. This is important in connection with the intensive use of nonsymmetric linear connections [1, 2] in different physical theories [6, 7].

This paper, which is a continuation of [8] and a revised version of [9], investigates the mentioned problem from the more general viewpoint of arbitrary derivations of the tensor algebra over a differentiable manifold [1, 2]. In it is proved the existence of special bases (or coordinates), called *normal*, in which the components of the derivations, as defined below, vanish along some path. In particular, our results are valid for linear connections. The normal bases are explicitly considered and the question when they are holonomic or anholonomic [2] is investigated.

As was pointed out in our previous work [8], where the above problems were solved in a neighborhood and at a point, the theorem of existence of normal bases is the right mathematical background for the consideration of the equivalence principle (cf. [7, 6]). The results of this paper outline the boundaries of validity of this principle along arbitrary paths in any gravitational theory based on derivations.

The paper is organized as follows. Sec. 2 contains some preliminary mathematical definitions and results. In Sec. 3 are investigated problems concerning normal frames for derivations along arbitrary vector fields. Sec. 4 and Sec. 5 deal with the same problems but for derivations along paths and a fixed vector field, respectively. The results are specialized for linear connections in Sec. 6. The paper closes Sec. 7 in which connections with the equivalence principle are made.

2 Mathematical preliminaries

For the explicit mathematical formulation of our problem, as well as for reference purposes, in this section we recall some facts concerning derivations of the tensor algebra over a manifold [8, 10, 1].

Let D be a derivation of the tensor algebra over a manifold M [1]. By [1, proposition 3.3 of chapter I] there exists a unique vector field X and a unique tensor field S of type $(1, 1)$ such that $D = L_X + S$. Here L_X is the Lie derivative along X [1] and S is considered as a derivation of the tensor algebra over M [1].

If S is a map from the set of C^1 vector fields into the tensor fields of

type (1,1) and $S : X \mapsto S_X$, then the equation $D_X^S = L_X + S_X$ defines a derivation of the tensor algebra over M for any C^1 vector field X [1]. Such a derivation will be called an S -derivation along X and denoted for brevity simply by D_X . An S -derivation is a map D such that $D : X \mapsto D_X$, where D_X is an S -derivation along X .

Let $\{E_i, i = 1, \dots, n := \dim(M)\}$ be a (coordinate or not [2, 11]) local basis (frame) of vector fields in the tangent bundle to M . It is holonomic (anholonomic) if the vectors E_1, \dots, E_n commute (do not commute) [2, 11]. Using the explicit action of L_X and S_X on tensor fields [1] one can easily deduce the explicit form of the local components of $D_X T$ for any C^1 tensor field T . In particular, the *components* $(W_X)_j^i$ of D_X are defined by

$$D_X(E_j) = (W_X)_j^i E_i. \quad (2.1)$$

Here and below all Latin indices, perhaps with some super- or subscripts, run from 1 to $n := \dim(M)$ and the usual summation rule on indices repeated on different levels is assumed. It is easily seen that $(W_X)_j^i := (S_X)_j^i - E_j(X^i) + C_{kj}^i X^k$ where $X(f)$ denotes the action of $X = X^k E_k$ on the C^1 scalar function f , as $X(f) := X^k E_k(f)$, and the C_{kj}^i define the commutators of the basic vector fields by $[E_j, E_k] = C_{jk}^i E_i$.

The change $\{E_i\} \mapsto \{E'_m := A_m^i E_i\}$, $A := [A_m^i]$ being a nondegenerate matrix function, implies the transformation of $(W_X)_j^i$ into (see (2.1)) $(W'_X)_l^m = (A^{-1})_i^m A_l^j (W_X)_j^i + (A^{-1})_i^m X(A_l^i)$. Introducing the matrices $W_X := [(W_X)_j^i]$ and $W'_X := [(W'_X)_l^m]$ and putting $X(A) := X^k E_k(A) = [X^k E_k(A_m^i)]$, we get

$$W'_X = A^{-1} \{W_X A + X(A)\}. \quad (2.2)$$

If ∇ is a linear connection with local components Γ_{jk}^i (see, e.g., [1, 11]), then $\nabla_X(E_j) = (\Gamma_{jk}^i X^k) E_i$ [1]. Hence, we see from (2.1) that D_X is a covariant differentiation along X iff

$$(W_X)_j^i = \Gamma_{jk}^i X^k \quad (2.3)$$

for some functions Γ_{jk}^i .

Let D be an S -derivation and X and Y be vector fields. The *torsion operator* T^D of D is defined as

$$T^D(X, Y) := D_X Y - D_Y X - [X, Y]. \quad (2.4)$$

The S -derivation D is *torsion free* if $T^D = 0$ (cf. [1]).

For a linear connection ∇ , due to (2.3), we have $(T^\nabla(X, Y))^i = T_{kl}^i X^k Y^l$ where $T_{kl}^i := -(\Gamma_{kl}^i - \Gamma_{lk}^i) - C_{kl}^i$ are the components of the torsion tensor of ∇ [1].

Mathematically the task of this work is to investigate the problem of when along a given path $\gamma : J \rightarrow M$, J being a real interval, there exist special frames $\{E'_i\}$, called *normal*, in which the components W'_X of an S-derivation D along some or all vector fields X vanish. In other words, we are going to solve equation (2.2) with respect to A under certain conditions, which will be specified below. Physically the solution of this problem corresponds to the investigation of the validity of the equivalence principle along paths.

3 Derivations along arbitrary vector fields

In this section we investigate the problem of existence and some properties of special bases $\{E'_i\}$ in which the components of a given S-derivation D_X along an *arbitrary vector field* X vanish along a path $\gamma : J \rightarrow M$, with J being an \mathbb{R} -interval. Such bases or frames will be called *normal along* γ . Note that $\{E'_i\}$ are supposed to be defined *in a neighborhood* of $\gamma(J)$, while the components of D vanish *on* $\gamma(J)$.

The S-derivation D is *linear along* γ if for all X in some (and hence in any) basis $\{E_i\}$, we have (cf. (2.3))

$$W_X(\gamma(s)) = \Gamma_k(\gamma(s))X^k(\gamma(s)) \quad (3.1)$$

for some matrix functions Γ_k defined on $\gamma(J)$. This means that (2.3) is valid for $x \in \gamma(J)$, but it may not be valid for $x \notin \gamma(J)$. Evidently, a linear connection is a linear derivation along any path γ .

Proposition 3.1 *An S-derivation D is linear along a path $\gamma : J \rightarrow M$ if and only if along γ there exists a normal frame for D , i.e. one in which the components of D_X along every vector field X vanish along γ (that is, on $\gamma(J)$).*

Proof. Let the derivation D be linear along γ , i.e. (3.1) is valid. Let us at first assume that γ is without self-intersections and that $\gamma(J)$ is contained in only one coordinate neighborhood U in which some local coordinate basis $\{E_i = \partial/\partial x^i\}$ is fixed.

Due to (2.2) we have to prove the existence of a matrix $A = [A_j^i]$ for which in the basis $\{E'_j = A_j^i E_i\}$ the equality $W'_X(\gamma(s)) = 0$ is fulfilled for every $X = X^k E_k$. Substituting (3.1) into (2.2), we see that the last equation is equivalent to

$$\Gamma_k(\gamma(s))A(\gamma(s)) + E_k(A) \big|_{\gamma(s)} = 0, \quad E_k = \partial/\partial x^k. \quad (3.2)$$

The general solution of this equation can be constructed as follows.

Let $V := J \times \cdots \times J$, where J is taken $n - 1$ times. Let us fix a one-to-one C^1 map $\eta : J \times V \rightarrow M$ such that $\eta(\cdot, \mathbf{t}_0) = \gamma$ for some fixed

$\mathbf{t}_0 \in V$, i.e. $\eta(s, \mathbf{t}_0) = \gamma(s)$, $s \in J$. This is possible iff γ is without self-intersections. In $U \cap \eta(J, V)$ we introduce coordinates $\{x^i\}$ by putting $(x^1(\eta(s, \mathbf{t})), \dots, x^n(\eta(s, \mathbf{t}))) = (s, \mathbf{t})$, $s \in J, \mathbf{t} \in V$. This, again, is possible iff γ is without self-intersections.

If we expand $A(\eta(s, \mathbf{t}))$ into a power series with respect to $(\mathbf{t} - \mathbf{t}_0)$, we find the general solution of (3.2) in the form

$$\begin{aligned} A(\eta(s, \mathbf{t})) = & \left\{ \mathbb{1} - \sum_{k=2}^n \Gamma_k(\gamma(s)) [x^k(\eta(s, \mathbf{t})) - x^k(\eta(s, \mathbf{t}_0))] \right\} \times \\ & \times Y(s, s_0; -\Gamma_1 \circ \gamma) B(s_0, \mathbf{t}_0; \eta) + \\ & + B_{kl}(s, \mathbf{t}; \eta) [x^k(\eta(s, \mathbf{t})) - x^k(\eta(s, \mathbf{t}_0))] [x^l(\eta(s, \mathbf{t})) - x^l(\eta(s, \mathbf{t}_0))]. \end{aligned} \quad (3.3)$$

Here: $\mathbb{1}$ is the unit matrix, $s_0 \in J$ is fixed, B is any nondegenerate matrix function of its arguments, the matrix functions B_{kl} are such that they and their first derivatives are bounded when $\mathbf{t} \rightarrow \mathbf{t}_0$, and $Y = Y(s, s_0; Z)$, with Z being a continuous matrix function of s , is the unique solution of the matrix initial-value problem [12, ch. IV, §1]

$$\frac{dY}{ds} = ZY, \quad Y|_{s=s_0} = \mathbb{1}, \quad Y = Y(s, s_0; Z). \quad (3.4)$$

Hence, a matrix A , and, consequently, a basis $\{E'_i\}$ with the needed property exist.

If $\gamma(J)$ does not lie in only one coordinate neighborhood, then, by means of the above described method, we can obtain local normal frames in different coordinate neighborhoods which form a neighborhood of $\gamma(J)$. From these local normal bases we can construct a global normal basis along γ . Generally this frame will not be continuous in the regions of intersection of two or more coordinate neighborhoods. For example, suppose for some γ there doesn't exist one coordinate neighborhood containing $\gamma(J)$ but there are two coordinate neighborhoods U' and U'' such that $\gamma(J) \subset U' \cup U''$. Then in U' and U'' there are (see above) normal bases $\{E'_i\}$ and $\{E''_i\}$ along γ for D_X for every X . So, a global normal basis $\{E_i^0\}$ in $U' \cup U''$ can be obtained by putting $E_i^0|_x = E'_i|_x$ for $x \in U'$ and $E_i^0|_x = E''_i|_x$ for $x \in U'' \setminus U'$ (note that $U'' \cap U'$ is not empty as γ is C^1 path).

Analogously, if γ has self-intersections, then on any 'part' of γ without self-intersections there exist local normal frames. From these frames can be constructed a global normal one along γ . (At the points of self-intersections of γ we can arbitrary fix these bases to be the ones obtained above for some fixed part of γ without self-intersections.)

Consequently, if D is linear along γ , then in a neighborhood of $\gamma(J)$ a basis $\{E'_i\}$ which is normal along γ exists for D_X for every vector field X .

Conversely, let us assume the existence of a frame $\{E'_i\}$ which is normal along γ , i.e. $W'_X = 0$ for every X . Fixing some basis $\{E_i\}$ such that

$E'_j = A_j^i E_i$, from (2.2) we find $(W_X A + X(A))|_{\gamma(s)} = 0$. Consequently $W_X(\gamma(s)) = -[(X(A))A^{-1}]|_{\gamma(s)}$ which means that the equation (3.1) holds for $\Gamma_k(\gamma(s)) = -[(E_k(A))A^{-1}]|_{\gamma(s)}$. ■

Proposition 3.2 *All frames which are normal along a path γ for an S -derivation, if any, are connected along γ by linear transformations whose coefficients are such that the action of the vectors from these bases on them vanish along γ (i.e. on $\gamma(J)$).*

Proof. If $\{E_i\}$ and $\{E'_i\}$ are normal frames, then we have $W'_X(\gamma(s)) = W_X(\gamma(s)) = 0$. So, from (2.2) follows $X(A)|_{\gamma(s)} = 0$ for every vector field $X = X^k E_k$, i.e. $E_k(A)|_{\gamma(s)} = 0$. ■

Proposition 3.3 *If along a path $\gamma : J \rightarrow M$ there is a local holonomic (on $\gamma(J)$) normal frame for some S -derivation D , then D is torsion free on $\gamma(J)$. Conversely, if D is torsion free on $\gamma(J)$ and a smooth (C^1) normal frame for D along γ exists, then all frames which are normal for D are holonomic along γ .*

Remark. In the second part of this proposition we demand the frames to be smooth. This is necessary as any holonomic basis is such. Besides, as we saw in the proof of proposition 3.1, if $\gamma(J)$ is not contained in only one coordinate neighborhood or if γ has self-intersection, then, generally, along γ there does not exist a continuous, even anholonomic, basis with the needed property. But on any piece of γ without self-intersection which lies in only one coordinate neighborhood a continuous, but maybe anholonomic, normal basis exists.

Proof. If $\{E'_i\}$ is a normal basis, i.e. $W'_X(\gamma(s)) = 0$ for every X and $s \in J$, then, using (2.4), we find $T^D(E'_i, E'_j)|_{\gamma(s)} = -[E'_i, E'_j]|_{\gamma(s)}$. Consequently $\{E'_i\}$ is holonomic at $\gamma(s)$, i.e. $[E'_i, E'_j]|_{\gamma(s)} = 0$, iff $0 = T^D(X, Y)|_{\gamma(s)} = X^i(\gamma(s))Y^j(\gamma(s))(T^D(E'_i, E'_j)|_{\gamma(s)})$ (see proposition 3.1 and (3.1)) for every vector fields X and Y , which is equivalent to $T^D|_{\gamma(J)} = 0$.

Conversely, let $T^D|_{\gamma(J)} = 0$. We have to prove that any basis $\{E'_i\}$ along γ in which $W'_X(\gamma(s)) = 0$ is holonomic at $\gamma(s)$, $s \in J$. The holonomicity at $\gamma(s)$ means $0 = [E'_i, E'_j]|_{\gamma(s)} = \left\{ (A^{-1})_k^l \left(E'_i(A_j^k) - E'_j(A_i^k) \right) E'_l \right\}|_{\gamma(s)}$. But (see proposition 3.1) the existence of $\{E'_i\}$ is equivalent to $W_X(\gamma(s)) = (\Gamma_k X^k)|_{\gamma(s)}$ for every X . These two facts, combined with (2.4), show that $(\Gamma_k)_j^i|_{\gamma(s)} = (\Gamma_j)_k^i|_{\gamma(s)}$. Using this and $(\Gamma_k A + \partial A / \partial x^k)|_{\gamma(s)} = 0$ (see the proof of proposition 3.1), we find $E'_j(A_i^k)|_{\gamma(s)} = -A_j^l A_i^m (\Gamma_l)_m^k|_{\gamma(s)} =$

$E'_i(A_j^k)|_{\gamma(s)}$. Therefore $[E'_i, E'_j]|_{\gamma(s)} = 0$ (see above), i.e. $\{E'_i\}$ is a holonomic normal frame on $\gamma(J)$. ■

It can be proved (see lemma 4.1 below) that for any path $\gamma : J \rightarrow M$ every frame $\{E_i^\gamma\}$ defined only on $\gamma(J)$ can locally be extended to a holonomic frame $\{E_i^h\}$ defined in a neighborhood of $\gamma(J)$ and such that $E_i^h|_{\gamma(J)} = E_i^\gamma$. In particular, this is true for the restriction $E_i^\gamma = E'_i|_{\gamma(J)}$ of the normal bases $\{E'_i\}$ considered above. But in the general case, the extended holonomic bases $\{E_i^h\}$ will not have the special property that $\{E'_i\}$ has.

4 Derivations along paths

Let $\gamma : J \rightarrow M$, J being an \mathbb{R} -interval, be a C^1 path and X be a C^1 vector field defined in a neighborhood of $\gamma(J)$ in such a way that on $\gamma(J)$ it reduces to the tangent vector field $\dot{\gamma}$, i.e. $X|_{\gamma(s)} = \dot{\gamma}(s)$, $s \in J$. We call the restriction on $\gamma(J)$ of an S-derivation D_X along X (S-)derivation along γ and denote it by \mathcal{D}^γ . Of course, \mathcal{D}^γ generally depends on the values of X outside $\gamma(J)$, but, as this dependence is insignificant for the following, it will not be written explicitly. So, if T is a C^1 tensor field in a neighborhood of $\gamma(J)$, then

$$(\mathcal{D}^\gamma T)(\gamma(s)) := \mathcal{D}_s^\gamma T := (D_X T)|_{\gamma(s)}, \quad X|_{\gamma(s)} = \dot{\gamma}(s). \quad (4.1)$$

It is easily seen that $\mathcal{D}_s^\gamma T$ depends only on the values of $T|_x$ for $x \in \gamma(J)$, but not on the ones for $x \notin \gamma(J)$. The operator \mathcal{D}^γ is a generalization of the usual covariant differentiation along curves (see [2, 3, 7] or Sect. 6).

When restricted to $\gamma(J)$, the components of D_X will be called components of \mathcal{D}^γ .

Proposition 4.1 *Along any C^1 path $\gamma : J \rightarrow M$ there exists a basis $\{E'_i\}$ in which the components of a given S-derivation \mathcal{D}^γ along γ vanish on $\gamma(J)$.*

Proof. Let us fix a basis $\{E_i\}$ in a neighborhood of $\gamma(J)$. We have to prove the existence of a transformation $\{E_i\} \rightarrow \{E'_j = A_j^i E_i\}$ such that $W_X|_{\gamma(J)} = 0$. By (2.2) this is equivalent to the existence of a matrix function $A = [A_j^i]$ satisfying along γ the equation $(A^{-1}(W_X A + X(A)))|_{\gamma(J)} = 0$, $s \in J$, or

$$\dot{\gamma}(A)|_{\gamma(s)} \equiv \frac{dA(\gamma(s))}{ds} = -W_X(\gamma(s))A(\gamma(s)) \quad (4.2)$$

as $X|_{\gamma(s)} = \dot{\gamma}(s)$. The general solution of this equation with respect to A is

$$\mathbf{A}(s; \gamma) = Y(s, s_0; -W_X \circ \gamma)B(\gamma), \quad (4.3)$$

where Y is the unique solution of the initial-value problem (3.4), $s_0 \in J$ is fixed, and $B(\gamma)$ is a nondegenerate matrix function of γ .

Let A be any matrix function with the property $A(x)|_{x=\gamma(s)} = \mathbf{A}(s; \gamma)$ for some s_0 and B . (E.g., using the notation of the proof of proposition 3.1, in any coordinate neighborhood in which γ is without self-intersections, we can put $A(\eta(s, \mathbf{t})) = Y(s, s_0; -W_X \circ \gamma)B(s_0, \mathbf{t}_0, \mathbf{t}; \gamma)$ for a fixed nondegenerate matrix function B .) Then it is easily seen that A carries out the needed transformation. Hence, the basis $\{E'_j = A^i_j E_i\}$ is the one looked for. ■

Proposition 4.2 *The normal frames along $\gamma : J \rightarrow M$ for \mathcal{D}^γ are connected by linear transformations whose coefficients on $\gamma(J)$ are constant or may depend only on γ .*

Proof. If $\{E_i\}$ and $\{E'_i\}$ are normal bases, then $W_X(\gamma(s)) = W'_X(\gamma(s)) = 0$, $X|_{\gamma(s)} = \dot{\gamma}(s)$. So, from (2.2) follows $\dot{\gamma}(A)|_{\gamma(s)} = dA(\gamma(s))/ds = 0$, i.e. $A(\gamma(s))$ is a constant or depends only on the map γ . ■

From propositions 4.1 and 4.2 we infer that the requirement for the components of \mathcal{D}^γ to vanish along a path γ determines the corresponding normal bases with some arbitrariness only on $\gamma(J)$ and leaves them absolutely arbitrary outside the set $\gamma(J)$. For this reason we speak about normal bases for \mathcal{D}^γ defined only on $\gamma(J)$.

Proposition 4.3 *Let the basis $\{E'_i\}$ defined on $\gamma(J)$ be normal for some S -derivation \mathcal{D}^γ along a C^1 path $\gamma : J \rightarrow M$. Let U be a coordinate neighborhood such that in $U \cap (\gamma(J)) \neq \emptyset$ the path γ is without self-intersections. Then there is a neighborhood of $U \cap (\gamma(J))$ in U in which $\{E'_i\}$ can be extended to a coordinate basis, i.e. in this neighborhood there exist local coordinates $\{y^i\}$ such that $E'_i|_{\gamma(s)} = \partial/\partial y^i|_{\gamma(s)}$.*

Remark 1. This proposition means that locally any normal basis for \mathcal{D}^γ on $\gamma(J)$ can be thought of as (extended to) a coordinate, and hence holonomic, one (see proposition 4.2). In particular, if γ is contained in only one coordinate neighborhood and is without self-intersections, then every normal frame on $\gamma(J)$ for \mathcal{D}^γ can be extended to a holonomic one (see the proof of proposition 4.2).

Remark 2. This result is independent of the torsion of the derivation \mathcal{D} which induces \mathcal{D}^γ . The cause for this is the condition $X|_{\gamma(s)} = \dot{\gamma}(s)$ in (4.1).

Proof. The proposition is a trivial corollary from the proof of proposition 4.1 and the following lemma.

Lemma 4.1 *Let the path $\gamma : J \rightarrow M$ be without self-intersections and such that $\gamma(J)$ is contained in some coordinate neighborhood U , i.e. $\gamma(J) \subset U$. Let $\{E'_i\}$ be a smooth basis defined on $\gamma(J)$, i.e. $E'_i|_{\gamma(s)}$ depends smoothly on s . Then there is a neighborhood of $\gamma(J)$ in U in which coordinates $\{y^i\}$ exist such that $E'_i|_{\gamma(s)} = \partial/\partial y^i|_{\gamma(s)}$, i.e. $\{E'_i\}$ can be extended in it to a coordinate basis. ■*

Proof of lemma 4.1. Let $\eta : J \times V \rightarrow U$, $V := J \times \cdots \times J$ ($n - 1$ times), be a C^1 one-to-one map such that $\eta(\cdot, \mathbf{t}_0) = \gamma$ for some fixed $\mathbf{t}_0 \in V$, i.e. $\eta(s, \mathbf{t}_0) = \gamma(s)$, $s \in J$ (cf. the proof of proposition 3.1). In the neighborhood $\eta(J, V) \subset U$ we introduce coordinates $\{x^i\}$ by putting $(x^1(\eta(s, \mathbf{t})), \dots, x^n(\eta(s, \mathbf{t}))) = (s, \mathbf{t})$, $s \in J$, $\mathbf{t} \in V$. Let the nondegenerate matrix $[A_j^i(s; \gamma)]$ define the expansion of $\{E'_i\}$ with respect to $\{\partial/\partial x^i\}$, i.e.

$$E'_j|_{\gamma(s)} = A_j^i(s; \gamma) \left(\frac{\partial}{\partial x^i} \Big|_{\gamma(s)} \right). \quad (4.4)$$

Define the functions $y^i : \eta(J, V) \rightarrow \mathbb{R}$ by

$$\begin{aligned} y^i(\eta(s, \mathbf{t})) := & x_0^i + \int_{s_0}^s (A^{-1})_1^i(u; \gamma) du + (A^{-1})_j^i(s; \gamma) [x^j(\eta(s, \mathbf{t})) - x^j(\gamma(s))] + \\ & + f_{jk}^i(s, \mathbf{t}; \gamma) [x^j(\eta(s, \mathbf{t})) - x^j(\gamma(s))] [x^k(\eta(s, \mathbf{t})) - x^k(\gamma(s))] \end{aligned} \quad (4.5)$$

where $s_0 \in J$ and $x_0 \in \eta(J, V)$ are fixed and the functions f_{jk}^i together with their first derivatives are bounded when $\mathbf{t} \rightarrow \mathbf{t}_0$. Then, because of $\eta(\cdot, \mathbf{t}_0) = \gamma$, we find

$$\frac{\partial y^i}{\partial x^j} \Big|_{\gamma(s)} = \frac{\partial y^i}{\partial x^j} \Big|_{\eta(s, \mathbf{t}_0)} = (A^{-1})_j^i(s; \gamma). \quad (4.6)$$

As $\det[A_j^i(s; \gamma)] \neq 0, \infty$, from (4.6) it follows that the transformation $\{x^i\} \rightarrow \{y^i\}$ is a diffeomorphism on some neighborhood of $\gamma(J)$ lying in U . So, in this neighborhood $\{y^i\}$ are local coordinates. The coordinate basic vectors on $\gamma(J)$ corresponding to them are (see (4.6) and (4.4))

$$\frac{\partial}{\partial y^j} \Big|_{\gamma(s)} = \left(\frac{\partial x^i}{\partial y^j} \Big|_{\gamma(s)} \right) \frac{\partial}{\partial x^i} \Big|_{\gamma(s)} = A_j^i(s; \gamma) \frac{\partial}{\partial x^i} \Big|_{\gamma(s)} = E'_j|_{\gamma(s)}.$$

Hence $\{y^i\}$ are the local coordinates we are looking for. ■

Lemma 4.1 has also a separate meaning: according to it any locally smooth basis defined on $\gamma(J)$ can locally be extended to a *holonomic* basis in a neighborhood of $\gamma(J)$. Evidently, such an extension can be done in an anholonomic way too. Consequently, the holonomicity problem for a basis defined only on $\gamma(J)$ depends on the way this basis is extended in a neighborhood of $\gamma(J)$.

5 Derivations along a fixed vector field

Results, analogous to the ones of Sect. 3, are true also for S-derivations D_X along a *fixed vector field* X (see Sect. 2), in other words for a fixed derivation. This case is briefly considered below.

Proposition 5.1 *The S-derivation D_X along a fixed vector field X is linear along a path $\gamma : J \rightarrow M$, i.e. (3.1) holds for that fixed X , iff along γ a normal frame $\{E'_i\}$ for D_X exists, i.e. one in which the components of D_X vanish on $\gamma(J)$.*

Proof. If (3.1) is valid for the given X , then by the proof of proposition 3.1, equation (3.2) has solutions A given by (3.3). Consequently in the basis $\{E'_j = A^i_j E_i\}$ we have $W'_X(\gamma(s)) = [A^{-1}(W_X A + X(A))]|_{\gamma(s)} = [(A^{-1}X^k)|_{\gamma(s)}][(\Gamma_k A + E_k(A))|_{\gamma(s)}] \equiv 0$. Conversely, if in $\{E'_j = A^i_j E_i\}$ we have $W'_X(\gamma(s)) = 0$, then due to (2.2) $(W_X A + X(A))|_{\gamma(s)} = 0$ is valid, i.e. $W_X(\gamma(s)) = \Gamma_k(\gamma(s))X^k(\gamma(s))$ for $\Gamma_k(\gamma(s)) = -[(E_k(A))A^{-1}]|_{\gamma(s)}$ for the fixed vector field X . ■

Evidently, infinitely many Γ_k 's can be found for which (3.1) holds for a fixed X . Consequently, for D_X with a fixed X there always exist normal frames along any path γ . These frames will be explicitly constructed elsewhere for any subset of M .

Proposition 5.2 *The normal bases along γ for D_X for a fixed X are connected by linear transformations whose matrices are such that the action of X on them vanishes on $\gamma(J)$.*

Proof. If in $\{E_i\}$ and $\{E'_j = A^i_j E_i\}$ we have $W_X(\gamma(s)) = W'_X(\gamma(s)) = 0$, then due to (2.2) $X(A)|_{\gamma(s)} = 0$ is valid with $A := [A^i_j]$, i.e. $X(A)|_{\gamma(J)} = 0$. ■

For a fixed vector field X the analogue of proposition 3.3 is, generally, not true. But if for D_X , X being fixed, (4.1) is valid on $\gamma(J)$, then we can construct a class of S-derivations $\{D\}$ whose components for every X are given by (3.1). Evidently, for these derivations proposition 3.3 holds. Thus we have proved

Proposition 5.3 *If along γ for D_X with a fixed X (3.1) is valid and there is a local holonomic (on $\gamma(J)$) normal frame along γ for D_X , then the above described derivations $\{D\}$ are torsion free on $\gamma(J)$. Conversely, if $\{D\}$ are torsion free on $\gamma(J)$ and there exists a smooth normal frame for D_X , then between them exist holonomic ones, but generally not all of them are such.*

6 The case of linear connections

In this section we apply the preceding results about normal frames to the special case of a linear connection ∇ .

Corollary 6.1 *For any linear connection ∇ there exists along every path $\gamma : J \rightarrow M$ a field of bases in which the components of ∇ vanish on $\gamma(J)$. These bases are connected with one another in the way described in proposition 3.2.*

Proof. This result is a consequence from (2.3), propositions 3.1 and 3.2 and their proofs; in the former of the proofs a basis with the needed property is explicitly constructed. ■

Corollary 6.2 *One, and hence any, basis for a linear connection ∇ which is smooth on $\gamma(J)$ and normal along a path $\gamma : J \rightarrow M$, is holonomic if and only if ∇ is torsion free on $\gamma(J)$.*

Remark. If γ is without self-intersections and $\gamma(J)$ lies in only one coordinate neighborhood, then there exist holonomic normal bases (coordinates) for ∇ on $\gamma(J)$ if ∇ is *torsion free* and vice versa, which is a well known fact [1, 2, 3, 11].

Proof. The statement follows from (2.3) and propositions 3.1 and 3.3. ■

Corollary 6.3 *Let ∇ be a torsion-free linear connection and the path $\gamma : J \rightarrow M$ be without self-intersections and lying in only one coordinate neighborhood. Then for ∇ there exist normal coordinates on $\gamma(J)$, or, equivalently, locally holonomic normal bases.*

Remark. This corollary reproduces a classical theorem that can be found, for instance, in [3] or in [2, ch. III, §8], in the latter of which references to original papers are given.

Proof. The result follows from corollaries 6.1 and 6.2. ■

Corollary 6.4 *Let $\frac{D}{ds}|_{\gamma} := \nabla_{\dot{\gamma}}$ be the covariant derivative associated with ∇ along the C^1 path $\gamma : J \rightarrow M$. Then on $\gamma(J)$ there exist normal frames for $\nabla_{\dot{\gamma}}$. They are obtained from one another by linear transformations whose coefficients are constant or depend only on γ . If γ is without self-intersections and $\gamma(J)$ lies in only one coordinate neighborhood, then in some neighborhood of $\gamma(J)$ all of these normal frames can be extended in a holonomic way.*

Proof. The statement follows from propositions 4.1, 4.2, and 4.3. ■

7 Conclusion

The above investigation shows that under sufficiently general conditions there exist, generally anholonomic, bases in which the components of a derivation of the tensor algebra over a differentiable manifold M vanish along a path $\gamma : J \rightarrow M$. These bases (frames) are called *normal*. When the derivations are along paths, then the corresponding normal bases always can be taken as holonomic (or coordinate) ones. These results generalize a series of analogous ones concerning linear connections and originating from [4].

A feature of the case along paths considered here is its independence of the derivation's curvature, which wasn't even introduced here. The cause

for this is the one dimensionality of the paths (curves) considered as submanifolds of M . In this connection it is interesting to consider the analogous problems on arbitrary submanifolds of M , which will be done elsewhere.

Now we shall consider briefly the relation of the results obtained in this paper with the equivalence principle [7, 6]. According to it the gravitational field strength, usually identified with the components of some linear connection, is transformable to zero at a point by an appropriate choice of the local (called normal, geodesic, Riemannian, inertial, or Lorentz) coordinates or reference frame (basis). So, from a mathematical point of view, the equivalence principle states the existence of local bases in which the corresponding connection's components vanish at a point. The results of this investigation show the strict validity of this statement along any path (curve). Hence, we can make the following three conclusions: (i) Any gravitational theory based on space-time with a linear connection is compatible with the equivalence principle along every path, i.e. in it there exist (local) inertial frames along paths. These frames are generally anholonomic, but under some (not very restrictive from a physical point of view) conditions on the paths (see lemma 4.1) there exist such holonomic frames of reference. (ii) In gravitational theories based on linear connections the equivalence principle along paths must not be considered as a principle (in a sense of an axiom) as it is identically fulfilled because of their mathematical background. (iii) If we want the equivalence principle along paths to be valid in gravitational theories based on some (class of) tensor derivations (cf. [10, Sect.V]), then this principle will select only the theories based on linear connections, i.e. only those in which it is identically satisfied. In fact, suppose the gravitational field strength to be locally identified with the components of a certain tensor derivation. The equivalence principle along paths requires the vanishment of the gravitational field strength along paths. So, this leads to the possibility to transform the components of the tensor derivation to zero along *any* path. By proposition 3.1 this implies the derivation to be linear along every path which is possible iff it is linear at every point, i.e. iff it is a linear connection.

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